5. Distribution Probability

Under similar and homogenous conditions, when an experiment s conducted repeatedly, we come a cross two kinds of situations :

(i) out come or result which is certain or,

(ii) outcome or result which is not certain, but may be one of various possibilities.

The experiment is named trial and results are named as events or cases. for example, act of tossing the coin is a trial, while getting a head or a tail is an event.

Mathematical Representation of Probability

Let b exhaustive, mutually exclusive and equally likely causes of which a are favourable to an event E. The probability (p) of happening of event E as $\frac{a}{b}$.

$$\therefore \quad P = \frac{\text{Number of favournable cases}}{\text{Total number of exhaustive cases}} = \frac{a}{b}$$

note that $\quad 0 \leqslant P \leqslant 1$

and $\quad$ a is events are favourable to event E.

$\quad$ b-a are events not favanrable to event E.

$\therefore$ Probability for not happening $\bar{E}$ is : $q = \frac{b-a}{b}$

$$q = \frac{b-a}{b} = 1 - \frac{a}{b} = 1 - P$$

$$\therefore \quad P + q = 1 \quad , \quad \text{also} \quad 0 \leqslant q \leqslant 1$$

$\therefore$ p is called probability of success of event E.

q is called probability of failure of event E.

If trails are repeated large number of times,

$$\therefore \quad p = P(E) = \lim_{b \to \infty} \frac{a}{b}$$

b is a total number of trails of which a are favourable

Notation :-

- $P(E)$ and $P(\bar{E})$ are probabilities for event E to happen or not to happen respectivily.
- $P(E_i + E_j)$ is probability for happening at least one of the events E_i or E_j.
- $P(E_i E_j)$ probability for happening both event E_i & E_j
- $P(E_i / E_j)$ conditional probability for event E_i to happen when event E_j has already happened.
- For n events $E_1, E_2, \ldots E_n$

$P(E_1 + E_2 + \cdots + E_n)$ = probability of occurance at least one of events $E_1, E_2, \ldots E_n$.

$P(E_1 E_2 \ldots E_n)$ = Probability of simultance occurance of all events $E_1, E_2, \ldots E_n$.

Always $\quad P(E) + P(\bar{E}) = 1$

Theorm 1 Probability of happening any one of the n mutually exclusive events is equal to sum of probabilities of the happening of separate events.

$$P(E_1 + E_2 + \cdots + E_n) = P(E_1) + P(E_2) + \cdots + P(E_n)$$

If two events not mutually exclusive

$$P(E_1 + E_2) = P(E_1) + P(E_2) - P(E_1 E_2)$$

Theorm 2 probability of simultaneous happening of two events E_1 and E_2 is equal to the product of E_1 and the conditional probability of E_2 given that E_1 already happened

$$P(E_1 E_2) = P(E_1) \cdot P(E_2/E_1)$$

r $P(E_1 E_2) = P(E_2) \cdot P(E_1/E_2)$

Corollary

i) If two events are independent then $P\left(\frac{E_1}{E_2}\right) = P(E_1)$, $P\left(\frac{E_2}{E_1}\right) = P(E_2)$

Thus, $P(E_1 E_2) = P(E_1) P(E_2)$

ii) If n events are independent, then,

$$P(E_1 E_2 \cdots E_n) = P(E_1) P(E_2) \cdots P(E_n)$$

) $P(E_1 + E_2 + \cdots + E_n) = 1 - P(\bar{E}_1 \bar{E}_2 \cdots \bar{E}_n)$

) If n events are independent and $P(E_i) = p_i$

$$P(E_1 + E_2 + \cdots + E_n) = 1 - P(\bar{E}_1 \bar{E}_2 \cdots \bar{E}_n)$$

$$= 1 - (1-p_1)(1-p_2) \cdots (1-p_n)$$

Bernoulli's theorm : If p denotes the probability of an events E, then the probability that it will occur in exactly r out of n trails is

$$^{n}C_r\, p^r q^{n-r} \quad \text{where } q = 1-p$$

and $$^{n}C_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots\times 2\times 1}, \quad \text{note that } {}^{n}C_n = 1,\ {}^{n}C_0 = 1$$

Example One bag contains 4 white ball and 2 black balls. Another bag contain 3 white and 5 black balls. If one ball is drawn from each bag. Find the probability that:
(i) both are white (ii) both are black (iii) one black and one whi

Solution
- probability of drawing one white ball from the first bag contain 6 balls is $\frac{4}{6} = \frac{2}{3}$ and drawing one black $\frac{2}{6} = \frac{1}{3}$.
- probability of drawing one white ball from the second bag contain 8 balls is $\frac{3}{8}$, and of one black ball is $\frac{5}{8}$.

Since drawing from two bags are independent. Apply the multipli cation theorm

(i) both are white

probability $= \frac{2}{3} \cdot \frac{3}{8} = \frac{1}{4}$

(ii) probability ; both are black $= \frac{1}{3} \cdot \frac{5}{8} = \frac{5}{24}$

(iii) - one black from the first bag and one white from the second bag is $\frac{1}{3} \cdot \frac{3}{8} = \frac{1}{8}$

- one black from the second and one white from the first bag is $\frac{5}{8} \times \frac{2}{3} = \frac{5}{12}$. As these events mutally exclusive the addition theorm gives probability of one black and one white ball is $1/8 + 5/12 = 13/24$.

Ex Tweleve coins are tossed together. Find the probability of getting 8, 9 or 10 heads in a single toss.

Sol. The probability of getting head in a single toss of single coin is $= \frac{1}{2}$ and of getting tail is $= \frac{1}{2}$

$\therefore p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$

Probability of 8 heads $= {}^{n}C_r \, p^r q^{n-r} = {}^{12}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4$

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$= \frac{12 \times 11 \times 10 \times 9 \times \cancel{8 \times 7 \times 6 \times 5}}{\cancel{8 \times 7 \times 6 \times 5} \times 4 \times 3 \times 2 \times 1} \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4 = 0.121$

probability of 9 heads $= {}^{12}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^3 = 0.0537$

probability of 10 heads $= {}^{12}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^2 = 0.0161$

As these events are mutually exclusive, the total probability $= 0.121 + 0.0537 + 0.0161 = 0.19$

Ex probability of hitting the target are $\frac{1}{2}$ for A, $\frac{2}{3}$ for B. and $\frac{3}{4}$ for C. If they all fire at the same target find.
(i) only one of them hit the target (ii) at least one hit the target.

Sol. $P(E_1) = \frac{1}{2}$ $P(E_2) = \frac{2}{3}$ $P(E_3) = \frac{3}{4}$

$\therefore P(\bar{E}_1) = \frac{1}{2}$ $P(\bar{E}_2) = \frac{1}{3}$ $P(\bar{E}_3) = \frac{1}{4}$

(i) only one of them hits the target حدوث واحد وعدم حدوث الباقية

Probability $= P(E_1 \bar{E}_2 \bar{E}_3) + P(\bar{E}_1 E_2 \bar{E}_3) + P(\bar{E}_1 \bar{E}_2 E_3)$

$= P(E_1) P(\bar{E}_2) P(\bar{E}_3) + P(\bar{E}_1) P(E_2) P(\bar{E}_3) + P(\bar{E}_1) P(\bar{E}_2) P(E_3)$

$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$

(ii) at least one of them hits the target

Probability $= 1 - P(\bar{E}_1 \bar{E}_2 \bar{E}_3) = 1 - P(\bar{E}_1) P(\bar{E}_2) P(\bar{E}_3)$

$= 1 - \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{23}{24}$

5-1 Binomial Probability Distribution

Consider a trail in which there are only two possible out-come, a success or failure. let (p) is probability of success, and $(q = 1-p)$ probability of failur. And let the trail repeated for (n) times. Then the probability of getting (x) success in (n) trails is :-

$0 \leq P \leq 1$

$x = 0, 1, 2, \cdots, n$

$$^{n}C_x \, p^x q^{n-x} = f$$

f is known as the probability density function of x.

and $$\sum f = \sum {}^{n}C_x \, p^x (1-p)^{n-x} = (p + \overline{1-p})^n = 1$$

Also,

① mean, $$\overline{X} = \frac{\sum f x}{\sum f} = \sum_{x=0}^{n} {}^{n}C_x \, p^x (1-p)^{n-x} \cdot x = np$$

Ex: if $p = 0.5$, then mean $= np = \frac{n}{2}$

② mode, is the value of (x) for which $f(x)$ is maximum

$\therefore$ mode $= np + p - 1$ (if n

$= np + p$ (if n

Ex: if $p = 0.5$, mode $= \frac{n}{2}$ if n even and $\frac{n-1}{2}$ and $\frac{n+1}{2}$ if n is odd.

③ standard deviation

$$\sigma^2 = \frac{\sum f x^2}{\sum f} - \overline{x}^2 = np(1-p)$$

$\therefore$ standard deviation $\sigma = \sqrt{np(1-p)}$

Ex For binomial distribution mean is 6 and standard deviation is $\sqrt{2}$. Find the two term of the distribution.

1. mean $= np = 6$, $\sigma = \sqrt{np(1-p)} = \sqrt{2}$

i.e $1-p = \frac{2}{6} = \frac{1}{3}$ $\Rightarrow$ $\therefore$ $p = \frac{2}{3}$ and $n = 9$

$\therefore$ $f(x) = {}^{n}C_x\, p^x (1-p)^{n-x}$

The first term $= f(0) = {}^{9}C_0 (\frac{2}{3})^0 (\frac{1}{3})^9 = (\frac{1}{3})^9$, ${}^{9}C_0 = 1$

The second term $= f(1) = {}^{9}C_1 (\frac{2}{3})^1 (\frac{1}{3})^8 = \frac{9}{1!} (\frac{2}{3})(\frac{1}{3})^8 = \frac{2}{2187}$

Ex An ordinary die is thrown four times, find the probability that:

(i) six not observed (ii) six obsened one

(iii) six obsened at least towic.

sol. binomial distribution with $n = 4$ $p = \frac{1}{6}$

i) $P(0) = {}^{4}C_0 (\frac{1}{6})^0 (1 - \frac{1}{6})^4 = 0.482$

ii) $P(1) = {}^{4}C_1 (\frac{1}{6})^1 (1 - \frac{1}{6})^3 = 0.386$

iii) $P(X \geq 2) = 1 - P(0) - P(1) = 1 - 0.482 - 0.386$

$= 0.132$

note that

$P(x) = {}^{n}C_x\, p^x q^{n-x} = 1$

$= P(0) + P(1) + {}^{n}C_2\, p^2 q^{n-2} = 1$

$\therefore$ ${}^{n}C_2\, p^2 q^{n-2} = 1 - P(0) - P(1)$

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5-2 Poisson Distribution of Probability

This is a limiting case of binomial distribution under the following assumptions:

(i) n, number of trails is increased, $\lim_{n\to\infty}$

(ii) p, the probability of success in a single trail is very small $p \to 0$

(iii) np is a finite constant, say $np = m$

Then, the Binomial distribution is:

$$P(X) = {}^{n}C_{x}\, p^{x} q^{n-X}$$

$$\therefore \lim_{\substack{n\to\infty \\ p\to 0}} P(x) = \frac{m^{x}}{x!} e^{-m}$$

Thus, Poisson distribution is: $f = \dfrac{m^{x} e^{-m}}{x!}$

And

① mean $\bar{X} = \dfrac{\Sigma f x}{\Sigma f} = \sum e^{-m} \dfrac{m^{x}}{x!} x = m$

② mode $= (m-1)$ or m

③ standard deviation

variance $\sigma^{2} = \dfrac{\Sigma f x^{2}}{\Sigma f} - (\bar{X})^{2} = m$

$\therefore \sigma = \sqrt{m}$

Ex (2%) of materials produced by a machine are defective. Find the probability that (100) materials are produced by a machine, there will be 0, 1, and more than 2 defective material.

Sol: $n = 100 \qquad p = 0.02 \quad , \therefore \quad np = 0.02 \times 100 = 2 = m$

$$\therefore \; p(0) = \frac{m^x e^{-m}}{x!} = \frac{m^0 e^{-2}}{0!} = e^{-2}$$

$$p(1) = \frac{m^1 e^{-2}}{1!} = 2e^{-2}$$

$$P(\geq 2) = 1 - p(0) - p(1) = 1 - 3e^{-2}$$

5.3 Normal Probability Distribution

Is a continuous distribution of a variable x with probability density function :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

note that :

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

let $t = \frac{x-m}{\sigma}$ $\quad dt = \frac{1}{\sigma} dx \Rightarrow dx = \sigma\, dt$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = 1$$

In generall, if

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} = \phi(-t)$$

then

$$\int_{-t}^{0} \phi(t)\,dt = \int_{0}^{t} \phi(t)\,dt$$

where $\int_{t_1}^{t_2} \phi(t)\,dt$ known as normal probability integral

Ex The mean weight of 500 student is 151 Ib, and the standard deviation is 15 Ib. Assume that the wt. are normally distributed, find how many students weigh between 120 and 155 lb.

Sol. mean $m = 151$, $\sigma = 15$

$$\therefore t = \frac{x-m}{\sigma} = \frac{x-151}{15}$$

when $x = 120 \Rightarrow t_1 = \dfrac{120-151}{15} = -2.06$

at $x = 155 \Rightarrow t_2 = \dfrac{155-151}{15} = 0.26$

$$\therefore \text{Probability Integral } \int_{t_1}^{t_2} \Phi(t)\, dt = \int_{-2.06}^{0.26} \Phi(t)\, dt$$

$$= \int_{-2.06}^{0.26} \Phi(t)\, dt = \int_{-2.06}^{0} \Phi(t)\, dt + \int_{0}^{0.26} \Phi(t)\, dt$$

$$= \int_{0}^{2.06} \Phi(t)\, dt + \int_{0}^{0.26} \Phi(t)\, dt = 0.4806 + 0.1051$$

$$= 0.5857 \quad \text{from table}$$

$\therefore$ number of students who weigh between 120 and 155 lb

is $= 0.5857 \times 500 = 293$